Topological Thinking

Introduction to topology

The field of *general topology* provides the machinery for rigorously dealing with the concept of *closeness* as well as determining *degrees of closeness*. The tools thus offered by general topology have significant and sundry musical implications. For example, if a melody x is deformed, resulting in melody x', is there some (ideally non-arbitrary) topological space in which x' qualifies as being 'similar' to x? And if x' is similar to x, how similar?

Other examples abound. Say that one wishes to instantiate a protoype of some kind (e.g. a melody, a gesture, an electronic sound, etc.), such that an instance may deviate with regards to a parameter or collection of parameters P from that prototype, by a specified 'degree of deformation'. Then if he topologizes (i.e., imposes a topological structure on) the set which contains (at least) the elements of the prototype, he can instantiate deviations of the prototype such that each instance remains within some subspace of the topology, as determined by a 'degree of similarity' to the prototype.

As Guerino Mazzola notes, "The cornerstone of topology is the concept of a neighborhood. We are given a set *Top* of 'points' of whatever nature and want to give an axiomatic account of what it means that we stay in a neighborhood of a selected point *x*. All we need is a minimum of properties of neighborhoods."[1] Mazzola then introduces the following axioms:

[1] Topos of Music, 277

Axioms (topological space) *Every point* $x \in Top$ *has a non-empty system* N_x *of subsets* $U \subset Top$*, called* neighborhoods of x*, such that:*

1. *For every neighborhood* U*,* $x \in U$.
2. *If* U *is a neighborhood of* x*, then any larger set* $V \supset U$ *is also a neighborhood of* x.
3. *There is a special neighborhood* V *of* x *such that* V *is a neighborhood of all its elements, i.e.,* $\forall y \in V(V \in N_y)$.
4. *The intersection of two neighborhoods* U, V *of* x *is a neighborhood of* x.[2]

The third axiom enables one to define open sets in a natural way. A subset O of Top is open if it's a neighborhood of all its points; thus V in axiom 3 is an open set. The benefit of thinking in terms of open sets is that it reduces the neighborhoods of a topological space to its 'core' neighborhoods, viz. the open sets.[3] Thinking in terms of open sets rather than neighborhoods will prove to be fruitful for later considerations.

Suppose that one were to list all open sets $O \in Top$ which contain the point x, and finds that each of these sets also contains z. Then according to the topology there is no means of distinguishing x from z, since no matter how much one localizes x (by moving to ever-smaller neighborhoods), he will still find x with z. In this case, we say that x *degenerates to* or is *degeneratively equal to* z, since, *as far as we can tell*, x is no different from z. We may define the relation between x and z as follows:

[2] Topos of Music, 277
[3] Ibid. 278

$$\neg\exists V \in N_x \not\ni z \rightarrow Degen(x, z)$$ [If there does not exist a neighborhood of x that does not contain z, then x *degenerates to* z]

We may think of the $Degen$ relation as determining whether or not x is 'maximally close' to z. Counter-intuitively, the $Degen$ relation is *not* (necessarily) symmetrical, but it *is* transitive. In terms of the symmetry condition, this means that if x is maximally close to z, z may not be maximally close to x. Whereas for transitivity, if x is maximally close to z and z is maximally close to y, then x is maximally close to y. The lack of symmetry is absurd according to natural language; for example, "a person sitting near you in a crowded subway will have the same feeling that you equally sit near this person".[4] The transitivity condition is likewise absurd: for if I'm close to a person and that person is close to someone else, it doesn't mean that I'm equally close to both of them.

The $Degen$ relation just outlined is unsatisfying because it only determines whether or not x is *maximally* close to z. But what about other degrees of closeness? For this we must have a systematic method for associating the open sets in Top with information relating to closeness. Fortunately we are in luck, but will need to introduce some more technical machinery, including the concept of a partially ordered set (poset) and a graded poset.

Partial orders

A partial order generalizes the idea of a total order. Inutitively, for a totally ordered set T, an element p either precedes, succeeds, or is equal to an element q. On the other hand, for a partially ordered set P, two elements p and q may or may *not* be in a relationship of

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[4] Ibid. 278

predecession, succession, or equality. Formally, a poset is a binary relation $\preccurlyeq$ over a set P satisfying the following axioms:

Axioms (poset) *For every a, b, c in P*

1. *$a \preccurlyeq a$ (reflexivity)*
2. *If $a \preccurlyeq b$ and $b \preccurlyeq a$, then $a = b$ (antisymmetry)*
3. *If $a \preccurlyeq b$ and $b \preccurlyeq c$, then $a \preccurlyeq c$ (transitivity)*

What interests us is how the open sets of Top can be represented by a poset P, where set-inclusion determines order. That is, if $U, V \in Top$ and $U \subset V$, then $U \prec V$ in P. Furthermore, a *graded* poset is a poset P equipped with a function $\rho: P \to \mathbb{N}$ from P to the natural numbers satisfying the following requirements:

Axioms (graded poset) *For $p, q \in P$*

1. If $\rho(p) < \rho(q)$ then $p \prec q$.
2. If q covers p then $\rho(q) = \rho(p) + 1$.

Viewing a topological space in terms of a graded poset of its open sets enables one to view the topology at different 'levels of refinement', like viewing a forest from the sky versus the ground. For the rank-function ρ associates to each open set a 'degree of resolution'.

However, not every poset is amenable to being graded. It may be the case that $a \prec b \prec x$, $c \prec x$, but $c \nprec a$ and $c \nprec b$. There is not a more abhorrent idea than being unable to assign grades to each open set of Top, as this would nullify our capacity to view Top at different levels of refinement. Indeed, one may grow disturbed at the very thought. … But he needn't worry, for we can prove that *every* topological space is compliant with a grading. To prove this, we simply need to demonstrate that the occurrence of N_5 – viz. the smallest non-modular lattice – is impossible in the poset-representation of the open sets of Top. If we can prove this, then by induction our proof can be extended to any other non-modular lattice, thus vitiating all anxiety about whether or not the open sets of Top form a graded poset. Before moving on to the proof however, I must introduce two axioms regarding the open sets of Top:

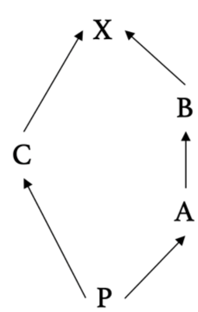
O1. If C is a collection of open sets in Top then so is its union, i.e., $\bigcup_{c \in C} c \in Top$.

O2. The intersection of any two sets in Top is also in Top.

Now the proof:

Proof

Let N_5 below denote the poset-representation of the open sets in Top.



$C \cup A = C \cup B = X$ (axiom *O1*)

$C \cap A = C \cap B = P$ (axiom *O2*)

$\big((C \cup A = C \cup B) \wedge (C \cap A = C \cap B)\big) \leftrightarrow A = B$

Contradiction. QED

Since we have proved that any topological space can be represented by a graded poset, we now know that a topological space has a well-defined number of different 'levels of resolution' at which it can be viewed. Therefore there is a spectrum of resolutions. At one pole exists maximal resolution, which equates to a maximally local view. In this case, we have much information about the locations of individual points, but little information about where those points exist relative to one another. On the other side of the spectrum exists minimal resolution. This equates to a maximally global view, in which case we have little information about the location of individual points, but more information about how those points relate to other, more distant points.

However, there are sometimes limits to the degrees of resolution of a topological space. For example, if one wishes to view a topological space at some degree of resolution r, then there may be only few points in the topology which are viewable at r. In other words, some points are more 'localizable' than others.

Revision of $Degen$ relation

In previous discussion we defined the $Degen$ relation as determining whether or not a point x is maximally close to a point z. But thanks to the idea of there being a spectrum of different degrees of 'localizability', we may revise the $Degen$ relation so that it determines *how close* x is to z. We can do this by checking *at which level(s)* of our graded poset x degenerates to z. It may be the case that x doesn't degenerate to z at some level $\rho = k$, but does at some level $\rho = n > k$. To specify the level we simply need to modify the $Degen$ relation discussed above. We can do this by making it a three-place relation taking as arguments the points to be compared and the level at which they are being compared. For example,

$$Degen(x, z, \rho_2)$$

states that x degenerates to z at level 2 (i.e., rank 2 of the graded poset). Letting $Parts_{\rho}(Top)$ denote the partition of the open sets of Top equivalent by ρ, the $Degen$ relation just outlined means that in $Parts_{\rho_2}(Top) \in Parts_{\rho}(Top)$ there is no set $U \subset Parts_{\rho_2}(Top)$ that contains x but not z.

Musical examples

One of the more obvious musical applications would be to impose a topological structure on the set of frequencies in the audio range. In such a scenario, closeness would be determined by how close two frequencies are on a logarithmic scale; e.g. C4 would be equally close to B4 as it would be to C#4. The neighborhood system of a given frequency *f* would consist therefore of frequency bandwidths of varying sizes (which contain *f*). If one were to choose one of these neighborhoods and let *f* vary rapidly within it, the result would be noise within the bandwidth determined by the size of the neighborhood. Thus moving to larger neighborhoods of *f* would result in more broadband noise, the limit being white noise.

A less obvious example would be the following. The underlying set is the set of twelve pitch-classes $\{C, C\#, \dots, B\}$, and the closest pitch-classes to a pitch-class x are those whose major scales have the most tones in common with an x-major scale. For example, if $x = C$, then C is closest to C, second closest to G and F, third closets to D and Bb, etc. So in this case the neighborhood system N_C of C would be the collection of sets

$$\{\{C\}, \{C, G, F\}, \{C, G, F, D, Bb\}, \dots, \{C, G, F, D, Bb, A, Eb, E, Ab, B, Db, F\#\}\}$$

An analog of noise in this topological space would be e.g. a melody M in the key of C, where each pitch-class in M is swapped with the corresponding pitch-class from a random key K in some neighborhood around C.

Of course there are many other ways in which one could express noise using such a topological space, or any other topological space for that matter. That is up to the imagination of the composer.

BIBLIOGRAPHY

Mazzola, Guerino. *The Topos of Music*. Basel, Switzerland 2002.